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Renormalization group treatment of the scaling properties of finite systems with subleading long-range interaction

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Abstract. The finite size behavior of the susceptibility, Binder cumulant and some even moments of the magnetization of a fully finite $O(n)$ cubic system of size L are analyzed and the corresponding scaling functions are derived within a field-theoretic ε-expansion scheme under periodic boundary conditions. We suppose a van der Waals type long-range interaction falling apart with the distance r as $r^{-(d+\sigma)}$, where $2 < \sigma < 4$, which does not change the short-range critical exponents of the system. Despite that the system belongs to the short-range universality class it is shown that above the bulk critical temperature T_c the finite-size corrections decay in a power-in-L, and not in an exponential-in-L law, which is normally believed to be a characteristic feature for such systems.

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1 Introduction

It is well known that the critical properties of a given bulk system depend on a small number of parameters like its dimensionality, the symmetry of the order parameter and the long-rangeness of the interaction in the system under consideration. If the Fourier transform of the interaction $v(\boldsymbol{q})$ has a small $|\boldsymbol{q}|$ expansion of the form

$$v(\boldsymbol{q}) = v_0 + v_2\boldsymbol{q}^2 + v_\sigma\boldsymbol{q}^\sigma + w(\boldsymbol{q}), \tag{1.1}$$

with $w(\boldsymbol{q})/\boldsymbol{q}^\sigma \to 0$ when $\boldsymbol{q} \to 0$ and $\sigma \geq 2$, then the thermodynamic critical behavior of the system is supposed to be like that of an entirely short-ranged system [1]. In the opposite case, when $\sigma < 2$ the critical behavior differs essentially [1,2] from that of the short-range system and is characterized by critical exponents that do depend on σ (below the corresponding upper critical dimension that is $d_u = 2\sigma$ in this case) [1]. On the basis of the above bulk picture one normally supposes that in the finite systems the same general property will take place: if $\sigma \geq 2$ the finite-size behavior will be that of the corresponding short-ranged finite-size systems [3], characterized by exponentially fast decay of the finite-size dependence of the thermodynamic quantities (at least when the critical region of the system is leaved in the direction towards higher temperatures; the low-temperature behavior depends on additional features like existence, or not, of a spin-wave excitations – Goldstone bosons). It turns out that the last is *not* true, at least for $2 < \sigma < 4$, and an evidence about that within the framework of the mean spherical model has been reported in [4]. For example, it has been demonstrated that the finite-size dependence of the susceptibility in such a system is given by ($2 < d < 4$, $2 < \sigma < 4$, $d + \sigma < 6$)

$$\chi(t, h; L) = L^{\gamma/\nu} Y(x_1, x_2, bL^{2-\sigma-\eta}), \tag{1.2}$$

or, equivalently,

$$\chi(t, h; L) = L^{\gamma/\nu}\left[Y^{sr}(x_1, x_2) + bL^{2-\sigma-\eta}Y^{lr}(x_1, x_2)\right], \tag{1.3}$$

where $x_1 = c_1 t L^{1/\nu}$, $x_2 = c_2 h L^{\Delta/\nu}$, and Y, Y^{sr} and Y^{lr} are universal functions (recall that $\eta = 0$ for the short-range spherical model). The quantities c_1, c_2 and b are nonuniversal constants, $t = (T - T_c)/T_c$ is the reduced temperature and h is a properly scaled external magnetic field. In the high-temperature, unordered phase, where $tL^{1/\nu} \to \infty$, one observes [4] that the long-range portion of the interaction between spin degrees of freedom gives rise to contributions of the order of $bL^{-(d+\sigma)}$. In other words the *subleading* long-range part of the interaction gives rise to a *dominant* finite-size dependence in this regime which is governed by a *power-in-L* law. More explicitly, one

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obtains $Y^{sr}(x_1, 0) \simeq Y_+^{sr} x_1^{-\gamma} + O(\exp(-\text{const. } x_1^{\nu}))$, while

$$Y^{lr}(x_1, 0) \simeq Y_1^{lr} x_1^{-2\gamma+\sigma\nu} + Y_2^{lr} x_1^{-2\gamma-d\nu}, \qquad (1.4)$$

when $x_1 \to \infty$ [4]. This asymptotic is supported from the existing both exact and perturbative results for models with *leading* long-range interaction [5–8]. Note that (1.4) implies for the temperature dependence of this corrections that

$$\chi(t, h; L) - \chi(t, h; \infty) \sim t^{-d\nu-2\gamma} L^{-(d+\sigma)}, \; tL^{1/\nu} \to \infty. \qquad (1.5)$$

In addition, let us note that the standard finite-size scaling [9–13] is usually formulated in terms of *only one* reference length, namely the bulk correlation length ξ. The main statements of the theory are that:

i) The only relevant variable in terms of which the properties of the finite system depend in the neighbourhood of the bulk critical temperature T_c is L/ξ.

ii) The rounding of the phase transition in a given finite system sets in when $L/\xi = O(1)$.

The tacit assumption is that all other reference lengths will lead *only to corrections* towards the above picture. As it is clear from equations (1.2–1.5) this is not the case in systems with subleading long-range interactions. This is an important class of systems. It contains all nonpolar fluids where the dominant interaction is supposed to be of van der Waals type, *i.e.* of the type given by equation (1.1) with $d = \sigma = 3$.

In fact a similar problem has been recently studied by Chen and Dohm [14–16]. They considered a field-theoretical model with short-range interactions and wavelength-dependent cutoff of fluctuations Λ. They observe corrections to the infinite system thermodynamic behavior going as an inverse power law in L that do depend also on $L\Lambda$ and not only on L/ξ. As it has been clarified in [4] the power law contributions to the finite size corrections result there from the interplay of two features of that model. The first is a sharp cutoff of fluctuations in momentum space and the second is the removal of all the terms beyond the q^2 one in (1.1), which has the effect of introducing an effective interaction that falls off as a power law in the separation between degrees of freedom. This power-law interaction leads immediately to power-law contributions to the finite size corrections.

Theoretically the critical properties of finite-size systems have been studied on the examples of exactly solvable models, by renormalization group calculations – both in the field-theoretical framework and in the real space, by conformal invariance and by numerical (mainly Monte Carlo) simulations. An essential part of these investigations is well described in a series of reviews [11–13,17,18].

The $O(n)$ models are the most often used examples on the basis of which one studies the scaling properties of finite-size systems. The best investigated cases are those of the $n = 1$ (Ising model) and the limit $n = \infty$, which includes the spherical model [12,13]. The last model is especially suitable for the investigation of its finite-size properties since it is exactly solvable for any d even in the presence of an external magnetic field. For $n \neq 1, \infty$ there are no exact results and the preferable analytical method for the derivation of the properties of the corresponding models (like XY, *i.e.* $n = 2$, and Heisenberg, *i.e.* $n = 3$) is that one of the renormalization group theory. An important amount of information for such systems is in addition derived by numerical simulations, normally *via* Monte Carlo methods. As a rule the investigations are concentrated on interactions of finite range. As examples of long-range interactions in addition to the equivalent neighbors the case of power-law decaying interactions have been considered. In the case of $\sigma < 2$ analytically only the finite-size scaling properties of the $n = \infty$ limit are well established. For finite n a limited number of recent numerical results [19–22], as well as few theoretical works [7,8,22,23] are available. The case of $\sigma < 2$ has been investigated in references [7,8,22] (under periodic boundary conditions). It has been found that, as for the bulk systems [1,24], the critical behavior depends on the small parameter $\varepsilon = 2\sigma - d$, where 2σ corresponds to the upper critical dimension in such systems [1,24]. The results are obtained in powers of $\sqrt{\varepsilon}$. The quantities of interest have been the shift of the critical coupling, the susceptibility and the Binder cumulant B at the critical temperature T_c [22,7] and above it [7,8] as a function of ε. It has been found that the numerical results obtained in [22] for the Ising model do not agree with the predicted (up to one loop order) behavior of B [7,22]. One is tempting to criticize the numerics, despite the authors claim that the method applied there suffice to account for the interaction of any spin with all others including with its own sequence of images under periodic boundary conditions (*i.e.* no truncation of the interaction has been enforced). In [21] one even reports disagreement with the well established theoretically fact that the critical exponents of the system do not depend on σ if $\sigma > 2$. Possible source of this disagreement are the finite-size corrections due to the long-range part of the interaction that cloud the short-range ones and that can be numerically essential for practically realizable sizes of the system.

In the present article we will consider the case of long-range power-law decaying interaction characterised by $\sigma > 2$ in its Fourier transform. As it was already mentioned above the recently obtained results for $n = \infty$ limit indicates that the well-spread opinion that such an interaction is uninteresting for the critical behavior of the finite system [3] is not fully correct. Here, following the method used in [7] we will generalize the results available for $n = \infty$ to the case of finite n. We will use ε-expansion technique up to one loop order in the interaction coupling. We will investigate the behavior of the Binder cumulant, susceptibility, and some more general even moments of the order parameter.

The plan of the article is as follows. In Section 2 we review, briefly, the φ^4-model with long-range interaction and discuss its bulk critical behavior. Section 3 is devoted to the explanation of the methods used here to achieve our analysis. We end the section with the computation of some thermodynamic quantities of interest. In Section 4 we discuss our results briefly. In the remainder of the paper we present details of the calculations of some formula used throughout the paper.

2 General considerations

In the vicinity of its critical point the Heisenberg model, with short as well as long-range interaction decaying in a power-law, is equivalent to the d-dimensional $\mathcal{O}(n)$-symmetric model

$$\beta\mathcal{H}\{\varphi\} = \frac{1}{2}\int_V \mathrm{d}^d\boldsymbol{x}\left[(\nabla\varphi)^2 + b\left(\nabla^{\sigma/2}\varphi\right)^2 + r_0\varphi^2 + \frac{1}{2}u_0\varphi^4\right], \tag{2.1}$$

where φ is a short-hand notation for the space dependent n-component field $\varphi(x)$, $r_0 = r_{0c} + t_0$ ($t_0 \propto T - T_\mathrm{c}$) and u_0 are model constants. V is the volume of the system and we assumed $k_\mathrm{B} = 1$. We note that the second term in the model denotes $\boldsymbol{q}^\sigma|\varphi(\boldsymbol{q})|^2$ in the momentum representation where the parameter $\sigma > 0$ (with $\sigma/2$ being noninteger) takes into account the contribution of the long-range interactions in the system. In (2.1) β is the inverse temperature. Here we will consider periodic boundary conditions. This means

$$\varphi(\boldsymbol{x}) = L^{-d}\sum_{\boldsymbol{q}}\varphi(\boldsymbol{q})\exp\left(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}\right), \tag{2.2}$$

where $\boldsymbol{q}$ is a discrete vector with components $q_i = 2\pi n_i/L$ ($n_i = 0, \pm1, \pm2, \cdots,\ i = 1, \cdots, d$) and a cutoff $\Lambda \sim a^{-1}$ (a is the lattice spacing). In this paper we are interested in the continuum limit, *i.e.* $a \to 0$. As long as the system is finite we have to take into account the following assumptions $L/a \to \infty$, $\xi \to \infty$ while ξ/L is finite.

The Hamiltonian (2.1) is, of course, well known in the literature. First, it has been used to investigate the critical behaviour of systems with reduced space dimensionality exhibiting phase transitions [1]. Let us recall that in such systems a phase transition can occur only if the interaction is long-ranged enough. The critical behaviour of the model depends strongly upon the nature of the interaction controlled by the parameter σ. With $\sigma \leq 2$ it has been used for detailed investigation of the critical behavior of $O(n)$ models including questions like the σ, d and n dependence of the critical exponents and critical amplitude ratios, as well as for calculation of their values, and for determining of the universal scaling functions of both the infinite, as well as of finite systems. In this case the critical exponents of the system are σ dependent. By increasing σ, a crossover from long-range critical behavior to short-range one takes place. The crossover happens at a point, which can be determined from general considerations (see for example page 71 of reference [25]). This 'critical' value of σ is given by $\sigma = 2 - \eta$, where η is the Fisher exponent for the short range model. When $\sigma > 2$ one usually considers the model as equivalent to $\sigma = 2$ case and omits the $b\left(\nabla^{\sigma/2}\varphi\right)^2$ term in the Hamiltonian, since it was believed that this term does not contribute to the critical behavior of the system. Indeed, in this case, the critical exponents do not depend on the parameter σ. As it was already mentioned, such a procedure can lead to incorrect results for finite-size systems. This was demonstrated in [4] on the example of $n = \infty$ model. In the current article we will demonstrate that the same remains true also for a finite n.

The investigation of the bulk critical behaviour of the model (2.1) for the case $\sigma > 2$ is achieved by considering the long range interaction as a perturbation to the short range one [26–28]. This allows the adaptation of the theory of Feynman diagrams to systems with subleading long range interaction. As a consequence the upper critical dimension remains unchanged by that interaction and the critical exponents are those of the model with pure short range interaction. The interested reader can find more details in references [26–28].

Before starting to explore the scaling properties of the field theoretical model (2.1) confined to a finite geometry and under periodic boundary conditions, we will give a brief heuristic derivation of the finite scaling hypothesis, based on the idea of renormalization group. Here we are interested in the continuum limit when the lattice spacing completely disappears. Using dimensional regularization the integrations over wave vectors of the fluctuations are convergent and are evaluated without cutoff. When some dimensions of the system are finite the integrals over the corresponding momenta are transformed into sums. Since the lattice spacing is taken to be zero, the limits of the sums still extend to infinity.

From general renormalization group considerations a multiplicatively renormalizable observable X, the susceptibility for example, will scale like

$$X[t, g, b, \mu, L] = \zeta(\rho)X\left[t(\rho), g(\rho), b(\rho), \mu\rho, L\right], \tag{2.3}$$

where $t = (T - T_\mathrm{c})/T_\mathrm{c}$ is the reduced temperature, g is a dimensionless coupling constant and L is the finite-size scale. The length scale μ is introduced in order to control the renormalization procedure. Here $b(\rho)$ is an irrelevant from RG point of view variable which mimics the influence of the subleading long range interaction on the critical behavior of the system. Equation (2.3) is obtained using the assumption that the size L of the system does not renormalize [29].

It is known (see, *e.g.*, [29]) that in the bulk limit, when $g(\rho)$ approaches the stable short-range fixed point g^* of the theory, we have

$$t(\rho) \approx t\rho^{1/\nu-2}, \quad \zeta(\rho) \approx \rho^{\gamma_x/\nu - p_x} \quad \text{and} \quad b(\rho) \approx b\rho^{\gamma_b - 2}, \tag{2.4}$$

where γ_x and ν are the bulk critical exponents measuring the divergence of the observable X and the correlation length, respectively, in the vicinity of the critical point and ρ is a scaling parameter. The exponent p_x is the dimension of the observable X, defined in equation (2.3).

The critical exponent $\gamma_b = 2 - \eta - \sigma$ [30]. Using dimensional analysis together with equation (2.3) one gets

$$X[t, g, b, \mu, L] = (\mu\rho)^{p_x}\zeta(\rho)X\left[t(\rho)(\rho\mu)^2, g(\rho), b(\rho)(\rho\mu)^2, 1, L/\mu\rho\right]. \quad (2.5)$$

Choosing the arbitrary parameter $\rho = L/\mu$, we obtain our final result for the scaling form of an observable X in the case, when there is subleading long-range interaction in the finite system

$$X[t, g, b, \mu, L] = L^{\gamma_x/\nu} f\left(tL^{1/\nu}, bL^{2-\sigma-\eta}\right). \quad (2.6)$$

Here the function $f(x)$ is a universal function of its arguments (after choosing in a proper way their scale factors). Note that equation (2.6) is the analog, for finite system, of the result obtained in [30]. In the remainder of this paper we will verify the scaling relation (2.6) in the framework of model (2.1).

3 Finite-size analysis

The method we will adopt here is widely used in the exploration of the scaling properties of finite systems in the vicinity of their critical point. It is based on the idea of using a mode expansion, *i.e.* one treats the zero mode of the order parameter, which is equivalent to the magnetization, separately from the higher modes ($q \neq 0$). The nonzero modes are treated perturbatively in combination with the loop expansion. The finite modes are traced over to yield an effective Hamiltonian for the zero mode:

$$\exp\left[-\mathcal{H}_{\text{eff}}\right] = \text{Tr}_{\phi_{q\neq 0}} \exp\left[-\mathcal{H}\left(\phi_{q=0}, \phi_{q\neq 0}\right)\right]. \quad (3.1)$$

After performing this operation one ends up with an effective Hamiltonian of the form (see Appendix A)

$$\mathcal{H}_{\text{eff}} = \frac{1}{2}L^d\left(R\phi^2 + \frac{1}{2}U\phi^4\right), \quad (3.2)$$

where the effective coupling constants are given by

$$R = r_0 + (n+2)u_0L^{-d}\sum_{\boldsymbol{q}\neq 0}\frac{1}{r_0 + \boldsymbol{q}^2 + b|\boldsymbol{q}|^\sigma}, \quad (3.3a)$$

$$U = u_0 - (n+8)u_0^2L^{-d}\sum_{\boldsymbol{q}\neq 0}\frac{1}{\left(r_0 + \boldsymbol{q}^2 + b|\boldsymbol{q}|^\sigma\right)^2}\cdot \quad (3.3b)$$

In the remainder of this paper we will compute, to the lowest order in $\varepsilon = 4 - d$, the effective coupling constants, with the initial coupling constants renormalized as in their bulk critical theory, since it has been shown that to the one loop order the renormalization of the finite theory is a consequence of the renormalization of the bulk one [29].

Simple dimensional analysis shows that the effective coupling constants should have the following scaling forms:

$$R = L^{\eta-2}f_R\left(tL^{1/\nu}, bL^{2-\sigma-\eta}\right)$$

and

$$U = L^{d-4+2\eta}f_U\left(tL^{1/\nu}, bL^{2-\sigma-\eta}\right), \quad (3.4)$$

for $t \gtrsim 0$, where f_R and f_U are scaling functions which are properties of the bulk critical point. They are analytic at $t = 0$. This is a consequence of the fact that only finite modes have been integrated out.

After evaluating the explicit forms of the functions f_R and f_U, we can deduce results for the different thermodynamic quantities and the expressions of their respective scaling functions.

In order to investigate the large scale physics of the finite system, one has to calculate thermal averages with respect to the new effective Hamiltonian defined in (3.2). They are related to the thermodynamic functions of the system under consideration. The averages of the field ϕ are defined by

$$\mathcal{M}_{2p} = \left\langle\left(\phi^2\right)^p\right\rangle = \frac{\int \mathrm{d}^n\phi\ \phi^{2p}\exp\left(-\mathcal{H}_{\text{eff}}\right)}{\int \mathrm{d}^n\phi \exp\left(-\mathcal{H}_{\text{eff}}\right)}\cdot \quad (3.5)$$

With the aid of the appropriate rescaling $\Phi = \left(UL^d\right)^{1/4}\phi$, we can transform the effective Hamiltonian into

$$\mathcal{H}_{\text{eff}} = \frac{1}{2}z\Phi^2 + \frac{1}{4}\Phi^4, \quad (3.6)$$

where the 'scaling variable' $z = RL^{d/2}U^{-1/2}$ is an important quantity which has been used in many occasions in the investigations of finite-size scaling in critical systems (see for example references [31,32]). Explicit expressions for some thermodynamic averages of the type (3.5) as well as their asymptotics are presented in Appendix B.

With the effective Hamiltonian (3.6), we obtain the general scaling relation

$$\mathcal{M}_{2p} = L^{-p(d-2+\eta)}\frac{L^{p(d-4+2\eta)/2}}{U^{p/2}}f_{2p}\left(RL^{2-\eta}\frac{L^{(d-4+2\eta)/2}}{U^{1/2}}\right) \quad (3.7)$$

for the momenta of the field ϕ. Having in mind equations (3.4), we can write down equation (3.7) in the following scaling form

$$\mathcal{M}_{2p} = L^{-p(d-2+\eta)}\mathcal{F}_{2p}(tL^{1/\nu}, bL^{2-\sigma-\eta}), \quad (3.8)$$

in agreement with the finite-size scaling predictions of (2.6). In equation (3.8), the functions $\mathcal{F}_{2p}(x)$ are universal.

All the measurable thermodynamic quantities can be obtained from the momenta $\mathcal{M}_{2p}$. For example the susceptibility is obtained from

$$\chi = \frac{1}{n}\int_V \mathrm{d}^dx\,\langle\varphi(\mathbf{x})\varphi(\mathbf{0})\rangle = L^{2-\eta}\mathcal{F}_2(tL^{1/\nu}, bL^{2-\sigma-\eta}). \quad (3.9)$$

Another quantity of importance for numerical analysis of the finite-size scaling theory is the Binder's cumulant defined by

$$\mathcal{B} = 1 - \frac{1}{3}\frac{\mathcal{M}_4}{\mathcal{M}_2^2}\cdot \tag{3.10}$$

In the remainder of this section we concentrate on the computation of the coupling constants R and U of the effective Hamiltonian (3.2) for the system with subleading long-range interaction decaying with the distance as a power law. As a consequence we will deduce results for the characteristic variable $z = RU^{-1/2}L^{2-\eta-\varepsilon/2}$, the susceptibility χ and the Binder's cumulant $\mathcal{B}$.

3.1 Computation of the effective coupling constants

The finite-size corrections to the coupling constant r_0 in the mode expansion reads

$$R = r_0 + (n+2)u_0 L^{-d}\sum_{\boldsymbol{q}\neq 0}\frac{1}{r_0 + \boldsymbol{q}^2 + b|\boldsymbol{q}|^\sigma}\cdot \tag{3.11}$$

One of the delicate problems in the finite size-scaling theory is the analysis of the sums appearing in the mathematical equations, which forms the basis of the investigation of the scaling as well as thermodynamic properties of the system under consideration. In our case this means that we have to find a way to evaluate the sum appearing in the right hand side of (3.11). In the absence of the long-range interaction term ($b = 0$) several methods have been developed in order to investigate this sum. When $b \neq 0$, *i.e.* in the presence of the non analytic term in $\boldsymbol{q}$, a step towards the solution of this problem has been made in reference [4]. It is based upon the idea that in the long distance physics one retains only those contribution to the behavior of the quantities involved that are associated with the effects of long-range fluctuations. In other words we will consider the leading behavior that is due to the small $\boldsymbol{q}$ contributions. Expanding in $\boldsymbol{q}$, we obtain

$$\begin{aligned} R = r_0 &+ (n+2)u_0 S_L(d, r_0, 2) \\ &- (n+2)u_0 b\left(1 + r_0\frac{\partial}{\partial r_0}\right)S_L(d, r_0, \sigma), \end{aligned}\tag{3.12}$$

where

$$S_L(d, r, \sigma) = \frac{1}{L^d}\sum_{\boldsymbol{q}\neq 0}\frac{|\boldsymbol{q}|^{\sigma-2}}{r + \boldsymbol{q}^2}\cdot \tag{3.13}$$

In order to evaluate the finite-size corrections to the bulk system we have to analyze the finite-size behavior of the function $S_L(d, r, \sigma)$. This is achieved by making use of the identity

$$\frac{q^{2p}}{r+q^2} = \int_0^\infty \exp[-(q^2+r)t]t^{-p}\gamma^*(-p, -rt)\mathrm{d}t, \quad p < 1, \tag{3.14}$$

where $\gamma^*(a, x)$ is a single-valued analytic function of a and x, possessing no finite singularities [33]

$$\begin{aligned} \gamma^*(a, x) &= \mathrm{e}^{-x}\sum_{n=0}^{\infty}\frac{x^n}{\Gamma(a+n+1)} \\ &= \frac{1}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{(-x)^n}{(a+n)n!}, \quad |x| < \infty. \end{aligned}\tag{3.15}$$

Identity (3.14) can be proven by integrating by parts the corresponding series representations of γ^*. Similar identity has been used in [23] for the investigation of the finite-size behavior of $\mathcal{O}(n)$ model system with a crossover from leading long-range interaction to the short-range case, *i.e.* $\sigma \to 2^-$.

With the help of this identity one obtains ($0 \leq p < 1$)

$$\begin{aligned} S_L(d, r, 2(p+1)) = \frac{1}{(2\pi)^d}\int \mathrm{d}\boldsymbol{q}\frac{\boldsymbol{q}^{2p}}{r+\boldsymbol{q}^2} \\ + L^{2-d-2p}I^p_{\text{scaling}}(rL^2, d), \end{aligned}\tag{3.16}$$

where

$$\begin{aligned} I^p_{\text{scaling}}(x, d) = (4\pi)^{p-1}\int_0^\infty \mathrm{e}^{-x\frac{u}{4\pi^2}}u^{-p}\gamma^*(-p, -\frac{x}{4\pi^2}u) \\ \times\left[\mathcal{A}^d(u) - \left(\frac{\pi}{u}\right)^{d/2} - 1\right]\mathrm{d}u, \end{aligned}\tag{3.17}$$

with

$$\mathcal{A}(u) = \sum_{k=-\infty}^{\infty}\mathrm{e}^{-k^2u} = \sqrt{\frac{\pi}{u}}\mathcal{A}\left(\frac{\pi^2}{u}\right)\cdot$$

Finally for the effective coupling constant R we obtain

$$\begin{aligned} R = r_0 &+ \frac{(n+2)u_0}{(2\pi)^d}\left(\int\frac{\mathrm{d}\boldsymbol{q}}{r_0+\boldsymbol{q}^2} - b\int \mathrm{d}\boldsymbol{q}\frac{|\boldsymbol{q}|^\sigma}{(r_0+\boldsymbol{q}^2)^2}\right) \\ &+(n+2)u_0L^{2-d}I^0_{\text{scaling}}\left(r_0L^2, d\right) \\ &-(n+2)u_0 b\left(1 + r_0\frac{\partial}{\partial r_0}\right)L^{4-d-\sigma}I^{\frac{\sigma-2}{2}}_{\text{scaling}}\left(r_0L^2, d\right). \end{aligned}\tag{3.18}$$

Now, we renormalize the theory by introducing the field theoretical renormalization constants, *i.e.* the scale field amplitude Z, the coupling constant renormalization Z_g, and Z_t – renormalizing the φ^2 insertions in the critical theory. This allows to replace the model bare constants r_0 and u_0 in the last equation by their renormalized counterparts trough:

$$t = ZZ_t^{-1}(r_0 - r_{0c}) \quad \text{and} \quad g = \mu^{-\varepsilon}Z^2Z_g^{-1}u_0S_d^{-1}, \tag{3.19}$$

where μ is a renormalization scale, which will be set equal to 1, $S_d = \frac{1}{2}(4\pi)^{d/2}\Gamma(d/2)$ is a phase space factor and [27,28]

$$Z = 1 + \mathcal{O}(g^2), \tag{3.20a}$$

$$Z_t = 1 + \frac{n+2}{\varepsilon}g + \mathcal{O}(g^2), \tag{3.20b}$$

$$Z_g = 1 + \frac{n+8}{\varepsilon}g + \mathcal{O}(g^2), \tag{3.20c}$$

are the usual renormalization amplitudes to one-loop order. According to references [27,28], the amplitudes given in (3.20) are the only relevant renormalization parameters necessary for the investigation of the bulk critical behaviour of the model (2.1) in the case of subleading long-range interaction, *i.e.* $\sigma > 2$. Up to the precision in which we are working we will see that these factors are indeed sufficient to renormalize the theory. Definitely, the question what will happen in higher orders is very important and interesting but it is out of the scope of the current investigation. Another potential difficulty comes from the fact that for $\sigma = 3$, the contributions proportional to $(k^2)^3$ will be of the same order as that produced by $(k^3)^2$, then it might happen that more sophisticated approach will be needed to account for higher-order contributions. Once again - this question is out of the scope of the current article.

Finally, using dimensional regularization, at the fixed point $g^* = \frac{\varepsilon}{n+8} + \mathcal{O}(\varepsilon^2)$ of the theory, in the case $d+\sigma < 6$ we obtain

$$
\begin{aligned}
RL^2 = {} & y\left(1+\frac{\varepsilon}{2}\frac{n+2}{n+8}\ln y\right) + \frac{\varepsilon}{4}\frac{n+2}{n+8}bL^{2-\sigma}\frac{(2+\sigma)\pi}{\sin\left(\pi\frac{\sigma}{2}\right)}y^{\sigma/2} \\
& + \varepsilon\frac{n+2}{n+8}S_4 I^0_{\text{scaling}}(y,4) \\
& - \varepsilon S_4\frac{n+2}{n+8}bL^{2-\sigma}\left(1+y\frac{\partial}{\partial y}\right)I^{\frac{\sigma-2}{2}}_{\text{scaling}}(y,4), \quad (3.21)
\end{aligned}
$$

where we have introduced the scaling variable $y = tL^{1/\nu}$ with $\nu^{-1} = 2 - \frac{n+2}{n+8}\varepsilon + \mathcal{O}\left(\varepsilon^2\right)$. Equation (3.21) shows that the effective coupling constant R has the scaling form predicted in equation (3.4). At this order, the exponent $\eta = 0$, and verifying the powers of η in this expression requires a higher order computation. In the particular case $b = 0$ from equation (3.21) we recover the result of reference [31].

When the system under consideration is confined to a finite geometry, instead of the coupling constant u_0, we have the shifted effective coupling constant U given by:

$$
U = u_0 - (n+8)u_0^2 L^{-d}\sum_{\boldsymbol{q}\neq 0}\frac{1}{\left(r_0+\boldsymbol{q}^2+b|\boldsymbol{q}|^{\sigma}\right)^2}\cdot \quad (3.22)
$$

Remark that the summand in the right hand side can be expressed as the derivative of the summand in the right hand side of equation (3.11) with respect to r_0. Consequently the result for the effective coupling constant U can be derived easily from that of R. Using that observation one gets

$$
\begin{aligned}
UL^{\varepsilon} = {} & \frac{\varepsilon}{n+8}S_4\left(1+\frac{\varepsilon}{2}(1+\ln y)\right) \\
& + \frac{\varepsilon^2}{n+8}\frac{bL^{2-\sigma}}{8}S_4\frac{\sigma\pi(\sigma+2)}{\sin\left(\pi\frac{\sigma}{2}\right)}y^{\frac{\sigma}{2}-1} \\
& + \frac{\varepsilon^2}{n+8}S_4^2\frac{\partial}{\partial y}I^0_{\text{scaling}}(y,4) - \frac{\varepsilon^2}{n+8}S_4^2 bL^{2-\sigma} \\
& \times\left(2\frac{\partial}{\partial y}+\frac{\partial^2}{\partial y^2}\right)I^{\frac{\sigma-2}{2}}_{\text{scaling}}(y,4), \quad (3.23)
\end{aligned}
$$

at the fixed point, in agreement with the scaling relations of equation (3.4). Equation (3.23) generalizes the results of reference [31] to the case when subleading long range interaction is taken into account.

Note that the effective coupling constant U has a finite limit at the critical point, *i.e.* in the limit $t \to 0$. Indeed, as the reduced temperature vanishes it is possible to use the expansion

$$
I^0_{\text{scaling}}(y,4) = I^0_{\text{scaling}}(0,4) + \frac{S_4^{-1}}{2}y\left(\mathcal{C}-\ln y\right) + \mathcal{O}(y^2), \quad (3.24)
$$

where

$$
\mathcal{C} = \int_0^{\infty}\frac{\mathrm{d}u}{u}\left[\exp\left(-\frac{u}{4\pi^2}\right) - \frac{u^2}{\pi^2}\mathcal{A}^4(u) + \frac{u^2}{\pi^2}\right] = 2.2064... \quad (3.25)
$$

After substitution of (3.24) in (3.23) the terms proportional to $\log y$ cancel, which shows that the coupling constant U is finite at $t = 0$. Whence, one gets (for $y \to 0$)

$$
\begin{aligned}
UL^{\varepsilon} = {} & \frac{\varepsilon}{n+8}S_4\left(1+\frac{\varepsilon}{2}\mathcal{C}\right) \\
& - \frac{\varepsilon^2}{n+8}S_4^2 bL^{2-\sigma}\left(2\frac{\partial}{\partial y}+\frac{\partial^2}{\partial y^2}\right)I^{\frac{\sigma-2}{2}}_{\text{scaling}}(y,4)\Big|_{y=0}, \quad (3.26)
\end{aligned}
$$

showing that it is possible to evaluate U at the critical point, *i.e.* it is safe now to set $y = 0$.

3.2 Some thermodynamic quantities

3.2.1 Binder's cumulant

In this subsection we are interested in the evaluation of the Binder's cumulant ratio, which plays a fundamental role in the investigation of the finite-size scaling theory by numerical means. Here we will give only the analytical expressions. Unfortunately there are no numerical simulation which can approve or not the results we obtain throughout this paper.

Close to the critical point, *i.e.* in the region $tL^{1/\nu} \ll 1$, we obtain for the Binder's cumulant ratio

$$
\begin{aligned}
\mathcal{B} = 1 - \frac{n}{12}\frac{\Gamma^2\left[\frac{1}{4}n\right]}{\Gamma^2\left[\frac{1}{4}(n+2)\right]}\Bigg\{ & 1 - z\left(\frac{\Gamma\left[\frac{1}{4}(n+6)\right]}{\Gamma\left[\frac{1}{3}(n+4)\right]}\right. \\
& \left. + \frac{\Gamma\left[\frac{1}{4}(n+2)\right]}{\Gamma\left[\frac{1}{4}n\right]} - 2\frac{\Gamma\left[\frac{1}{4}(n+4)\right]}{\Gamma\left[\frac{1}{4}(n+2)\right]}\right) + z^2 \\
& \times\left(\frac{\Gamma\left[\frac{1}{4}(n+6)\right]\Gamma\left[\frac{1}{4}(n+2)\right]}{\Gamma\left[\frac{1}{4}(n+4)\right]\Gamma\left[\frac{1}{4}n\right]} + 3\frac{\Gamma^2\left[\frac{1}{4}(n+4)\right]}{\Gamma^2\left[\frac{1}{4}(n+2)\right]} - n - 1\right) \\
& + \mathcal{O}\left(z^3\right)\Bigg\}\cdot \quad (3.27)
\end{aligned}
$$

The cumulant $\mathcal{B}$ is a function of the variable z, which is itself a function of the scaling variable y. So, a knowledge of a final expression for the function z, which appears in the all thermodynamic quatities, is enough to deduce the value of the Binder's Cumulant. At the fixed point, we obtain (for $y \ll 1$)

$$z^*(y) \equiv \left.\frac{RL^2}{\sqrt{UL^\varepsilon}}\right|_{\text{fixedpoint}} = \sqrt{\frac{n+8}{\varepsilon S_4}}$$
$$\times \left[y - y\frac{\varepsilon}{4}\left(1 - \frac{n-4}{n+8}\ln y\right) + \frac{3n}{n+8}\frac{\varepsilon}{16}\frac{(2+\sigma)\pi}{\sin\left(\pi\frac{d}{\sigma}\right)}y^{\sigma/2} \right.$$
$$+\frac{n+2}{n+8}\varepsilon S_4\left(I^0_{\text{scaling}}(y,4)\right.$$
$$\left.-bL^{2-\sigma}\left(1+y\frac{\partial}{\partial y}\right)I^{\frac{\sigma-2}{2}}_{\text{scaling}}(y,4)\right)$$
$$-\frac{1}{2}\varepsilon S_4 y\left(\frac{\partial}{\partial y}I^0_{\text{scaling}}(y,4)\right.$$
$$\left.\left.-bL^{2-\sigma}\left(2\frac{\partial}{\partial y}+\frac{\partial^2}{\partial y^2}\right)I^{\frac{\sigma-2}{2}}_{\text{scaling}}(y,4)\right)\right]. \quad (3.28)$$

This expression shows that the Binder's Cumulant $\mathcal{B}$ has the required scaling form. At the critical point, *i.e.* at $y = 0$, we get

$$z^*(0) = -\sqrt{\varepsilon}\frac{4\sqrt{2}}{\pi}\frac{n+2}{\sqrt{n+8}}$$
$$\times\left[\ln 2 + bL^{2-\sigma}(2\pi)^{\sigma-2}(1-4^{\frac{\sigma}{2}-1})\zeta\left(1-\frac{\sigma}{2}\right)\zeta\left(2-\frac{\sigma}{2}\right)\right]. \quad (3.29)$$

This result is obtained with the help of the formula [34]

$$\int_0^\infty \mathrm{d}u u^{1-\nu}\left[\mathcal{A}^4(u) - 1 - \left(\frac{\pi}{u}\right)^2\right] =$$
$$8(1-4^{1-\nu})\pi^{2(1-\nu)}\Gamma(\nu)\zeta(\nu-1)\zeta(\nu), \quad \nu \neq 0, 2. \quad (3.30)$$

Equation (3.29) is a generalization of the result of [31] obtained for the model with pure short range forces. Note that the form of the expansion in terms $\sqrt{\varepsilon}$ is kept but the coefficient is altered and now it is a function the parameter σ controlling the long-range interaction.

Now we turn our attention to the behavior of Binder's cumulant ratio in the limit $z \gg 1$. In this case we obtain

$$\mathcal{B} = 1 - \frac{1}{3}\left(1+\frac{2}{n}\right)\left[1 - \frac{2}{z^2} + \mathcal{O}\left(\frac{1}{z^4}\right)\right], \quad (3.31)$$

wherefrom one has $\mathcal{B}_n(\infty) = \frac{2}{3}(1-1/n)$. This result corresponds to a n-dimensional Gaussian distribution for n independent components $\Phi_1, \cdots, \Phi_n$ of the vector variable Φ. For such a distribution it is easy to show that $M_2 = n\langle\Phi_i^2\rangle$, and $M_4 = n\langle\Phi_i^4\rangle + n(n-1)\langle\Phi_i^2\rangle^2$, where Φ_i is any of the components of the vector Φ, and $\langle\cdots\rangle$ means average with respect to one-component Gaussian distribution $\mathcal{G}_1$. Having in mind that for $\mathcal{G}_1$ $\langle\Phi_i^4\rangle = 3\langle\Phi_i^2\rangle^2$, one directly obtains that $B_n = \frac{2}{3}(1-1/n)$, in a full agreement with the above renormalization group result. Obviously, all limiting values lie in the interval from $\mathcal{B} = 0$ (Ising model, $n = 1$) to $\mathcal{B} = 2/3$ (spherical model, $n = \infty$).

3.2.2 Magnetic susceptibility

The system we consider here is confined to a fully finite geometry. In this case it cannot exhibit a true phase transition, *i.e.* the thermodynamic functions are not singular. In the vicinity of the critical temperature, which corresponds to the region $y \ll 1$, the susceptibility behaves like

$$\chi = \frac{2}{n}\frac{L^2}{\sqrt{UL^\varepsilon}}\frac{\Gamma\left[\frac{1}{4}(n+2)\right]}{\Gamma\left[\frac{n}{4}\right]}$$
$$\times\left\{1 - z\left(\frac{n}{4}\frac{\Gamma\left[\frac{n}{4}\right]}{\Gamma\left[\frac{1}{4}(n+2)\right]} - \frac{\Gamma\left[\frac{1}{4}(n+2)\right]}{\Gamma\left[\frac{n}{4}\right]}\right)\right.$$
$$\left.+z^2\left(\frac{1-n}{4} + \frac{\Gamma^2\left[\frac{1}{4}(n+2)\right]}{\Gamma^2\left[\frac{n}{4}\right]}\right) + \mathcal{O}(z^3)\right\}\cdot \quad (3.32)$$

The susceptibility in this case is analytic at $t = 0$. This is a consequence of the analyticity of the effective coupling constants R and U. In order to get the final result for the susceptibility one has to replace R and U by their respective expressions. After performing this we find that χ has an expansion in powers of $\sqrt{\varepsilon}$. This results is valid as long as we are concerned by the case $d + \sigma < 6$. Once we have $d + \sigma = 6$, a $\ln L$ will appear in the expression of the susceptibility. The source of this $\ln L$ is coming from equation (C3) for the coupling constant U at the critical point (see Appendix C). This is an extension to finite n, by means of perturbation method, of the result obtained in reference [4] for the spherical model.

In the region corresponding to $y \gg 1$, we have

$$\chi = \frac{1}{R}\left[1 - \frac{n+2}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right)\right]\cdot \quad (3.33)$$

Substituting the effective coupling constants R and U by their respective expressions from equations (3.21, 3.23), and using the asymptotic expansions of I^p_{scaling} derived in [4], we get

$$\chi = \chi_\infty\left[1 - \varepsilon\frac{n+2}{n+8}\frac{S_4}{y}bL^{2-\sigma}\left(C_{4,\frac{\sigma-2}{2}}y^{-2} - \frac{y^{\sigma/2}}{4S_4}\frac{\sigma+2}{\sin\pi\sigma/2}\right)\right], \quad (3.34)$$

where

$$C_{d,p} = -\frac{(1+p)4^{1+p}}{\pi^{d/2}}\frac{\Gamma(1+p+d/2)}{\Gamma(-p)}\sum_{\boldsymbol{k}\neq 0}\frac{1}{k^{d+2(p+1)}}\cdot \quad (3.35)$$

Expression (3.34) for the susceptibility shows that it has the form given by the scaling hypothesis (3.9). It demonstrates also that in this regime the critical properties of the system are dominated by the bulk critical behavior, with finite-size corrections in powers of L.

4 Discussion

In the present article we have investigated the finite-size scaling behavior of a fully finite $O(n)$ system with periodic boundary conditions and in the presence of a long-range interaction that does not alter the short-range exponents of critical its critical behavior. The small $|\boldsymbol{q}|$ expansion of the Fourier transform of the interaction $v(\boldsymbol{q})$ is supposed to be of the form

$$v(\boldsymbol{q}) = v_0 + v_2\boldsymbol{q}^2 + v_\sigma \boldsymbol{q}^\sigma + w(\boldsymbol{q}), \tag{4.1}$$

with $w(\boldsymbol{q})/\boldsymbol{q}^\sigma \to 0$, when $\boldsymbol{q} \to 0$ and $2 < \sigma < 4$. In the real d-dimensional space one can think about interactions decaying as $r^{-(d+\sigma)}$. This is an important class of interactions that include also van der Waals type interactions.

For such a system, in the present article we have demonstrated that all the even moments of the magnetization $\mathcal{M}_{2p}$, including the susceptibility, can be written in the form

$$\mathcal{M}_{2p} = L^{-p(d-2+\eta)}\mathcal{F}_{2p}(tL^{1/\nu}, bL^{2-\sigma-\eta}), \tag{4.2}$$

(see Eqs. (B4, 3.21, 3.23, 3.32, 3.34). Note that one has two scaling variables needed in order to describe in a proper way the finite size behavior of these quantities. A special attention has been paid to two important quantities: the Binder's Cumulant and the susceptibility.

In the region $tL^{1/\nu} \gg 1$ away from the critical point we obtained for the Binder's cumulant ratio the expression

$$\mathcal{B} = 1 - \frac{1}{3}\left(1 + \frac{2}{n}\right), \tag{4.3}$$

with finite size correction falling off in a power law. The above result corresponds to a n-dimensional Gaussian distribution for n independent components of the vector variable. Obviously, all the values lie in the interval from $\mathcal{B} = 0$ (Ising model, $n = 1$) to $\mathcal{B} = 2/3$ (spherical model, $n = \infty$).

For the susceptibility, when $tL^{1/\nu} \gg 1$, one has (see Eq. (3.34))

$$\chi = \chi_\infty\left[1 - \varepsilon\frac{n+2}{n+8}\frac{S_4}{y}bL^{2-\sigma}\left(C_{4,\frac{\sigma-2}{2}}y^{-2} - \frac{y^{\sigma/2}}{4S_4}\frac{\sigma+2}{\sin\pi\sigma/2}\right)\right]. \tag{4.4}$$

One observes that in this regime the susceptibility approaches its bulk value not in an exponential-in-L, as it is usually believed to be the case for systems with short-range critical exponents, but in a power-in-L way. The last goes beyond the standard formulation of the finite-size scaling, but is completely consistent with the intrinsic large-distance power-law behavior of the correlations in systems with long-range interactions (see, *e.g.* [35] and references cited therein).

Since $\eta = O(\varepsilon^2)$ in $O(n)$ short-range models, we were unable to verify the predicted dependence of the scaling functions on η, which requires calculations up to second order of ε, while we have retained only corrections up to the first order in ε. We hope to return to this problem in the future.

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Appendix A: Construction of the effective Hamiltonian

Let us start with the bare Hamiltonian (2.1)

$$\mathcal{H}\left\{\varphi\right\} = \frac{1}{2}\int_V \mathrm{d}^d x\left[(\nabla\varphi)^2 + b\left(\nabla^{\sigma/2}\varphi\right)^2 + r_0\varphi^2 + \frac{1}{2}u_0\varphi^4\right], \tag{A1}$$

where the spatial integration is over a system of linear extent L in each of its d dimensions. The partition function is given by

$$\mathcal{Z} = \int \mathcal{D}\varphi \exp(-\mathcal{H}). \tag{A2}$$

Following reference [32], we spilt the field

$$\varphi(x) = \phi + \varSigma \tag{A3}$$

into a mode independent part ϕ, which defines the magnetization, and a part depending on the nonzero modes $\varSigma = L^{-d}\sum_{\mathbf{q}\neq\mathbf{0}}\varphi(\boldsymbol{q})\exp\left(\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}\right)$. For further calculation we introduce the auxiliary Hamiltonian

$$\mathcal{H}_0\left\{\phi\right\} = L^d\left[\frac{1}{2}r_0\phi^2 + \frac{1}{4}u_0\phi^4\right]. \tag{A4}$$

and we treat the rest of the Hamiltonian by using perturbation theory. Within this approximation the partition function reads

$$\mathcal{Z} = \int \mathcal{D}\phi \exp\left(-H_0(\phi) - \overset{0}{\varGamma}(\phi)\right), \tag{A5}$$

where

$$\overset{0}{\varGamma} = -\ln\int \mathcal{D}\varSigma \exp(-\mathcal{H}(\phi,\varSigma) + \mathcal{H}_0(\phi)). \tag{A6}$$

Writing the difference between the bare Hamiltonian (A1) and the auxiliary Hamiltonian (A4) in the form

$$\mathcal{H}\left\{\phi,\varSigma\right\} - \mathcal{H}_0\left\{\phi\right\} = \frac{1}{2}\int_V \mathrm{d}^d x\left[\left(r_0 + 3u_0\phi^2\right)\varSigma^2 + (\nabla\varSigma)^2 + b\left(\nabla^{\sigma/2}\varSigma\right)^2\right] \tag{A7a}$$

$$+\frac{1}{2}u_0\int_V \mathrm{d}^d x\left[2\phi\varSigma^3 + \frac{1}{2}\varSigma^4\right] \tag{A7b}$$

keeping in mind that the additional term involving $\int_V \mathrm{d}^d x\varSigma$ vanishes one gets, after some straightforward calculations, including the evaluation of the integrals over the field $\varSigma$,

$$\overset{0}{\varGamma}(\phi^2) = \frac{1}{2}\sum_{\boldsymbol{q}\neq 0}\ln[r_0 + \boldsymbol{q}^2 + b\boldsymbol{q}^\sigma] + \frac{1}{2}(n+2)u_0\phi^2 L^d\mathcal{S}_1(r_0,L) - \frac{1}{4}(n+8)u_0^2\phi^4 L^d\mathcal{S}_2(r_0,L) + \cdots, \tag{A8}$$

where

$$\mathcal{S}_m(r_0, L) = L^{-d} \sum_{\boldsymbol{q}\neq 0} \frac{1}{(r_0 + \boldsymbol{q}^2 + b\boldsymbol{q}^\sigma)^m}, \qquad \text{(A9)}$$

and the dots represent terms with higher order in ϕ.

Substituting expression (A9) into that of the partition function (A5), we end up with the final expression for the effective Hamiltonian

$$\mathcal{H}_{\text{eff.}} = \frac{1}{2} L^d \left[R\phi^2 + \frac{1}{2} U \phi^4 \right], \qquad \text{(A10)}$$

where the effective coupling constants are given by

$$R = r_0 + (n+2) u_0 L^{-d} \sum_{\boldsymbol{q}\neq 0} \frac{1}{r_0 + \boldsymbol{q}^2 + b|\boldsymbol{q}|^\sigma}, \qquad \text{(A11a)}$$

$$U = u_0 - (n+8) u_0^2 L^{-d} \sum_{\boldsymbol{q}\neq 0} \frac{1}{\left(r_0 + \boldsymbol{q}^2 + b|\boldsymbol{q}|^\sigma\right)^2} \cdot \qquad \text{(A11b)}$$

These are the finite-size corrections to the bulk coupling constants r_0 and u_0, which are necessary for the evaluation of various thermodynamic quantities.

Appendix B: Finite-size scaling behavior of the even moments of the order parameter

By definition the $2p$th moment of the order parameter of an $O(n)$ model is given by

$$\langle M_{2p} \rangle_n = \frac{\int_0^\infty \mathrm{d}\Phi \Phi^{2p} \mathrm{e}^{-\frac{1}{2} L^d \left[R\Phi^2 + \frac{1}{2} U \Phi^4\right]}}{\int_0^\infty \mathrm{d}\Phi \mathrm{e}^{-\frac{1}{2} L^d \left[R\Phi^2 + \frac{1}{2} U \Phi^4\right]}} \cdot \qquad \text{(B1)}$$

Changing the variable of integration to $\varphi = (UL^d)^{1/4}\Phi$ and by introducing the scaling variable $z = RL^{d/2}/\sqrt{U}$ the above expression can be rewritten in the form

$$\langle M_{2p} \rangle_n = \left(UL^d\right)^{-\frac{p}{2}} \frac{\int_0^\infty \mathrm{d}\varphi \varphi^{2p+n-1} \mathrm{e}^{-\frac{1}{2} z\varphi^2 - \frac{1}{4}\varphi^4}}{\int_0^\infty \mathrm{d}\varphi \varphi^{n-1} \mathrm{e}^{-\frac{1}{2} z\varphi^2 - \frac{1}{4}\varphi^4}} \cdot \qquad \text{(B2)}$$

Using the identity [36]

$$\int_0^\infty x^{\nu-1} \mathrm{e}^{-\beta x^2 - \gamma x} \mathrm{d}x = (2\beta)^{-\nu/2} \Gamma(\nu) \exp\left(\frac{\gamma^2}{8\beta}\right) D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right), \qquad \text{(B3)}$$

where $D_p(z)$ are the parabolic cylinder functions, the above expression can be rewritten in a very simple form

$$\langle M_{2p} \rangle_n = \left(UL^d/2\right)^{-\frac{p}{2}} \frac{\Gamma[p+n/2]}{\Gamma[n/2]} \frac{D_{-p-n/2}(z/\sqrt{2})}{D_{-n/2}(z/\sqrt{2})} \cdot \qquad \text{(B4)}$$

Using now the asymptotics of $D_p(z)$ [36] it is straightforward to obtain the asymptotic behavior of the above moments for i) $z \gg 1$ and ii) $z \ll 1$.

i) $z \gg 1$. Then one has

$$\langle M_{2p} \rangle_n = \left(UL^d/4\right)^{-\frac{p}{2}} \frac{\Gamma[p+n/2]}{\Gamma[n/2]} z^{-p} \times \left[1 - \frac{p(n+p+1)}{z^2} + O\left(\frac{1}{z^4}\right)\right] \cdot \qquad \text{(B5)}$$

ii) $z \ll 1$. For this case the corresponding result is

$$\begin{aligned}\langle M_{2p} \rangle_n = {} & \left(\frac{UL^d}{4}\right)^{-\frac{p}{2}} \frac{\Gamma[\frac{p}{2}+\frac{n}{4}]}{\Gamma[\frac{n}{4}]} \\ & \times \left[1 + z\left(\frac{\Gamma[\frac{1}{2}+\frac{n}{4}]}{\Gamma[\frac{n}{4}]} - \frac{\Gamma[\frac{p}{2}+\frac{n}{4}+\frac{1}{2}]}{\Gamma[\frac{n}{4}+\frac{p}{2}]}\right)\right. \\ & + z^2 \left(\frac{\Gamma[\frac{1}{2}+\frac{n}{4}]}{\Gamma[\frac{n}{4}]} \left(\frac{\Gamma[\frac{1}{2}+\frac{n}{4}]}{\Gamma[\frac{n}{4}]} - \frac{\Gamma[1+\frac{n}{4}]}{2\Gamma[\frac{n}{4}+\frac{1}{2}]}\right.\right. \\ & \left.\left.\left. - \frac{\Gamma[\frac{1}{2}+\frac{n}{4}+\frac{p}{2}]}{\Gamma[\frac{n}{4}+\frac{p}{2}]}\right) + \frac{\Gamma[1+\frac{n}{4}+\frac{p}{2}]}{2\Gamma[\frac{n}{4}+\frac{p}{2}]}\right) + O(z^3)\right].\end{aligned} \qquad \text{(B6)}$$

For the susceptibility ($p = 1$) the above expression can be written in the following very simple form

$$\langle M_2 \rangle_n = a_n + z b_n + z^2 c_n + O(z^3), \qquad \text{(B7)}$$

where $a_n = \Gamma(\frac{n}{4}+\frac{1}{2})/\Gamma(\frac{n}{4})$, $b_n = a_n^2 - \frac{n}{4}$, $c_n = a_n(b_n + 1/4)$.

From (B5) it follows that the asymptotic behavior of the Binder cumulant is

$$B_n(z) = 1 - \frac{1}{3}\left(1 + \frac{2}{n}\right)\left[1 - \frac{2}{z^2} + O\left(\frac{1}{z^4}\right)\right], \qquad \text{(B8)}$$

wherefrom one has $B_n(\infty) = \frac{2}{3}(1 - 1/n)$.

Appendix C: Finite-size results for the physically important case: $d + \sigma = 6$

In this Appendix we will report some results for the important case $d + \sigma = 6$, which models the van der Waals type potential. Note that because of the condition $d + \sigma = 6$ one now has only one independent variable, *i.e.* setting $d = 4 - \varepsilon$ directly leads to $\sigma = 2 + \varepsilon$. If one performs now ε-expansion on the σ-dependent terms one will in fact change the spectrum of the system from such one, where $q^\sigma = q^{2+\varepsilon}$ is considered as a perturbation to the short-range contribution (proportional to q^2), to one in which q^σ is replaced by $q^2 + \varepsilon q^2 \ln q$, *i.e.* where the long-range portion of the interaction will represent already a leading-order term. This is not the type of systems we are interested in. Therefore, in order to avoid this problem, in all the calculations below we perform ε-expansion only on the d-dependent terms and retain the full ε-dependence in all terms where it is stemming from the σ-dependence of the quantities involved. Following this way of acting we obtain that in the case $d+\sigma=6$ the expression (3.21) for R transforms into

$$\begin{aligned}RL^2 = y\left(1+\frac{\varepsilon}{2}\frac{n+2}{n+8}\ln y\right) - \varepsilon\frac{n+2}{n+8}bL^{-\varepsilon}y(\ln y - 2\ln L)\\ +\varepsilon\frac{n+2}{n+8}S_4\left[I^0_{\text{scaling}}(y,4) - bL^{-\varepsilon}\left(1+y\frac{\partial}{\partial y}\right)\right.\\ \left.\times I^{\frac{\varepsilon}{2}}_{\text{scaling}}(y,4)\right],\end{aligned} \quad \text{(C1)}$$

showing that there is an additional $\ln L$ correction to the finite-size scaling theory. Definitely, keeping the terms proportional to $L^{-\varepsilon}$ one goes beyond the precision kept in the remaining part of the above equation. In accordance with the remarks made above note that wile one does not perform an expansion of $L^{-\varepsilon}$ in terms of ε all the terms in (C1) proportional to $L^{-\varepsilon}$ are simply corrections to scaling. But, once one performs that expansion, because of the $\ln L$ proportionality, these terms produce a leading-order contribution, which is quite unphysical. We believe that this is an artifact of the ε expansion. Such a procedure (keeping the full ε-dependence in some expressions) has been used in reference [37] in the analysis of the scaling properties of quantum systems at low temperatures. We hope that the above problems can be removed by performing, *e.g.*, a field theoretical method based on minimal renormalization at fixed space dimensionality [14,38]. This is out of the scope of the current article.

For the coupling constant U, instead of (3.23) one obtains

$$\begin{aligned}UL^{\varepsilon} = &\frac{\varepsilon}{n+8}S_4\left[1+\frac{\varepsilon}{2}(1+\ln y)\right]\\ &-\frac{\varepsilon^2}{n+8}bL^{-\varepsilon}S_4\left[\ln y - 2\ln L\right]\\ &+\frac{\varepsilon^2}{n+8}S_4^2\left[\frac{\partial}{\partial y}I^0_{\text{scaling}}(y,4)\right.\\ &\left.-bL^{-\varepsilon}\left(2\frac{\partial}{\partial y}+\frac{\partial^2}{\partial y^2}\right)I^{\frac{\varepsilon}{2}}_{\text{scaling}}(y,4)\right].\end{aligned} \quad \text{(C2)}$$

In this limit an additional $\ln L$ correction shows up. This expression is finite in the limit $y=0$. At the critical point it transforms into

$$\begin{aligned}UL^{\varepsilon} = \frac{\varepsilon}{n+8}S_4\left[1+\frac{\varepsilon}{2}\mathcal{C}\right]\\ -\frac{\varepsilon^2}{n+8}bS_4\left[\frac{1}{2}\mathcal{C}-2\ln L - 1\right] - \frac{5}{2}\frac{\varepsilon^2}{n+8}bL^{-\varepsilon}\zeta(3).\end{aligned} \quad \text{(C3)}$$

The explicit appearance of $\ln L$ will affect the result of the susceptibility, which will depend upon an additional $\ln L$ at the critical point $T=T_{\text{c}}$.

At the fixed point, for the 'characteristic' variable z, we obtain

$$\begin{aligned}z^*(y) \equiv &\left.\frac{RL^2}{\sqrt{UL^{\varepsilon}}}\right|_{\text{fixedpoint}}\\ = &\sqrt{\frac{n+8}{\varepsilon S_4}}\left\{y - y\frac{\varepsilon}{4}\left(1-\frac{n-4}{n+8}\ln y\right)\right.\\ &+\varepsilon bL^{-\varepsilon}\frac{n+2}{n+8}y + \frac{1}{2}\varepsilon bL^{-\varepsilon}\frac{4-n}{n+8}y(\ln y - 2\ln L)\\ &+\frac{n+2}{n+8}\varepsilon S_4\left[I^0_{\text{scaling}}(y,4)\right.\\ &\left.-bL^{-\varepsilon}\left(1+y\frac{\partial}{\partial y}\right)I^{\frac{\varepsilon}{2}}_{\text{scaling}}(y,4)\right]\\ &-\frac{1}{2}\varepsilon S_4 y\left[\frac{\partial}{\partial y}I^0_{\text{scaling}}(y,4)\right.\\ &\left.\left.-bL^{-\varepsilon}\left(2\frac{\partial}{\partial y}+\frac{\partial^2}{\partial y^2}\right)\right]I^{\frac{\varepsilon}{2}}_{\text{scaling}}(y,4)\right\},\end{aligned} \quad \text{(C4)}$$

A comparison between (3.28) obtained for the case $d+\sigma<6$ and (C4) shows that an additional $\ln L$ appears in the expression of the variable $z(y)$, however this does not alter the result (3.29) for $z^*(0)$, *i.e.* $z(y)$ evaluated at the critical point $T=T_{\text{c}}$. In this case the term proportional to $\ln L$ vanishes as we take the limit $y\to 0$. The result (C4) shows, in this way, that the Binder Cumulant at the critical point does not depend on $\ln L$.

Far away from criticality the susceptibility (3.34) found for the case $d+\sigma<6$ turns into

$$\chi = \chi_{\infty}\left[1+\varepsilon b\frac{n+2}{n+8}S_4^{-1}\left(\ln\chi_{\infty}+5\varepsilon S_4\zeta(3)y^{-3}\right)\right]. \quad \text{(C5)}$$

for the case $d+\sigma=6$. Remark that the susceptibility conserves the same features as that of the case discussed in the body of the paper.

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